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Noisy Zhang-Dynamics (ZD) Method for Genesio Chaotic (GC) System Synchronization: Elegant Analyses and Unequal-Parameter Extension

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***Abstract*—This paper handles the noise-free or noisy synchronization control of Genesio chaotic (GC) system. To do so, Zhang dynamics (ZD) method is presented and exploited, and thus the ZD controllers, noise-free or noisy, are theoretically researched. Firstly, the presented ZD controller for GC system synchronization with no noise perturbation is analyzed, and the synchronization error as a whole (i.e., in the form of vector norm) between the drive GC system and the response GC system converges globally exponentially to zero. Secondly, the presented ZD controller for GC system synchronization with noise perturbation is analyzed as well, and detailed theoretical analyses (i.e., proofs) and results show that the synchronization error as a whole (i.e., in the form of vector norm) converges globally to a small bound of error. So, the ZD controllers provided in this paper (including the ones with unequal parameters) are not only simple and effective but also quite robust for the GC system synchronization.**

***Keywords*—noisy; Zhang dynamics (ZD); Genesio chaotic (GC) system; synchronization; analyses; unequal parameters.**

I. Preliminaries and Controllers

The differential equations of drive Genesio chaotic (GC) system [1, 2] are

$$\begin{cases} \dot{x}_1(t) = x_2(t) \in \mathbb{R}, \\ \dot{x}_2(t) = x_3(t) \in \mathbb{R}, \\ \dot{x}_3(t) = -cx_1(t) - bx_2(t) - ax_3(t) + x_1^2(t) \in \mathbb{R}, \end{cases} \quad (1)$$

where $x_1(t) \in \mathbb{R}$, $x_2(t) \in \mathbb{R}$ and $x_3(t) \in \mathbb{R}$ denote state variables, and the parameters $a \in \mathbb{R}^+$, $b \in \mathbb{R}^+$ and $c \in \mathbb{R}^+$ are positive real constants, with $ab < c \in \mathbb{R}^+$.

To synchronize with drive GC system (1), the response GC system with control input $u(t) \in \mathbb{R}$ added is

$$\begin{cases} \dot{y}_1(t) = y_2(t) \in \mathbb{R}, \\ \dot{y}_2(t) = y_3(t) \in \mathbb{R}, \\ \dot{y}_3(t) = -cy_1(t) - by_2(t) - ay_3(t) \\ \qquad\qquad + y_1^2(t) + u(t) \in \mathbb{R}. \end{cases} \quad (2)$$

The synchronization purpose is that the trajectories of the response GC system (2) globally converge to those of the drive GC system (1). The synchronization errors are thus defined between the drive GC system (1) and the response GC system (2) as follows:

$$\begin{cases} e_1(t) = y_1(t) - x_1(t) \in \mathbb{R}, \\ e_2(t) = y_2(t) - x_2(t) \in \mathbb{R}, \\ e_3(t) = y_3(t) - x_3(t) \in \mathbb{R}. \end{cases} \quad (3)$$

For the convenience of simply and better presenting, $e_i = e_i(t) \in \mathbb{R}$, $x_i = x_i(t) \in \mathbb{R}$, $y_i = y_i(t) \in \mathbb{R}$ with $i = 1, 2, 3$, and $u = u(t) \in \mathbb{R}$. Thus, on the basis of (2) and (3), we have the differential equations of the synchronization errors as the following ones:

$$\begin{cases} \dot{e}_1 = e_2 \in \mathbb{R}, \\ \dot{e}_2 = e_3 \in \mathbb{R}, \\ \dot{e}_3 = -ce_1 - be_2 - ae_3 + (x_1 + y_1)e_1 + u \in \mathbb{R}. \end{cases} \quad (4)$$

Besides, $\boldsymbol{e} = [e_1, e_2, e_3]^{\mathrm{T}} \in \mathbb{R}^3$ stands for the synchronization error vector. The basic purpose of the research is to design a controller $u \in \mathbb{R}$ so that the response system (2) is synchronized with the drive system (1), requiring that $\|\boldsymbol{e}\|_2 \in \mathbb{R}^+$ gradually converges to zero, or $\lim_{t\to+\infty} \sup \|\boldsymbol{e}\|_2 \in \mathbb{R}^+$ is a very little positive constant (near zero) in practice.

By ZD method [3–6] with its fundamental novelties discussed mainly in [6], we firstly design a simple, effective and robust ZD controller with equal parameters for the GC system synchronization. Without or with noise perturbation, the robust ZD controller can both synchronize the response system (2) with the drive system (1). Besides, the detailed theoretical analyses are given as one main contribution of the paper in the next section, extended to unequal parameters.

Now, for synchronizing the response GC system (2) with the drive GC system (1), the ZD controller is designed and established via the following simple, direct and straightforward ZD steps and formulas.

$$z_1 = e_1 = y_1 - x_1 \in \mathbb{R}. \quad (5)$$

$$\dot{e}_1 = -\lambda e_1 \in \mathbb{R}. \quad (6)$$

$$z_2 = \dot{e}_1 + \lambda e_1 = e_2 + \lambda e_1 \in \mathbb{R}. \quad (7)$$

$$\dot{z}_2 = -\lambda z_2 \in \mathbb{R}. \quad (8)$$

$$e_3 + 2\lambda e_2 + \lambda^2 e_1 = 0 \in \mathbb{R}. \quad (9)$$

$$z_3 = e_3 + 2\lambda e_2 + \lambda^2 e_1 \in \mathbb{R}. \quad (10)$$

$$\dot{z}_3 = -\lambda z_3 \in \mathbb{R}. \quad (11)$$

Finally, on the basis of equations (4) and (11), the ZD controller $u \in \mathbb{R}$ is obtained as the following:

$$u = (c-\lambda^3)e_1 + (b-3\lambda^2)e_2 + (a-3\lambda)e_3 - (x_1+y_1)e_1 \in \mathbb{R}. \quad (12)$$

Note that, in practice, noises are everywhere. In general, noises are expressed as additive noise $\delta(t) \in \mathbb{R}$. In this paper, we suppose that these noises are also bounded, i.e., $|\delta(t)| \leqslant \delta_{\max} \in \mathbb{R}^+$ with $\delta_{\max}$ as a positive bound. Thus, the proposed controller (12) under the pollution of noises in practice now becomes

$$\tilde{u} = (c-\lambda^3)e_1 + (b-3\lambda^2)e_2 + (a-3\lambda)e_3 - (x_1+y_1)e_1 + \delta(t). \quad (13)$$

II. Theoretical Analyses on Controller u or $\tilde{u}$

In this section, the performance analyses (including theoretical results) of the ZD controller about noise perturbation are shown. In the first part, the controller u (12) for synchronizing the GC system is analyzed, and the synchronization error as a whole (i.e., in a vector norm) between the drive GC system (1) and the response GC system (2) is proved to globally exponentially converge to zero. In the second part, controller $\tilde{u}$ (13) for the synchronization of drive GC system (1) is analyzed, and the detailed theoretical analyses (including theoretical results) show that the synchronization error as a whole (i.e., in a vector norm) globally converges to a small error bound. In other words, the effectiveness and robustness of the ZD controller for synchronization of Genesio chaotic system control are proved through the detailed theoretical analyses as follows (please see [7, 8] and references therein).

A. ZD Controller Without Noise Perturbation

The performance analyses (including theoretical results) on noise-free ZD controller u (12) for synchronization of the GC system (1) are presented now and here, where u, again, denotes the control input of response GC system (2). As the preliminary, the following lemma is also provided [9].

Lemma 1: For $\tilde{\alpha} > 0 \in \mathbb{R}$, $\tilde{\beta} > 0 \in \mathbb{R}$ and $t \geq 0 \in \mathbb{R}$, there exist $\alpha > 0 \in \mathbb{R}$ and $\beta > 0 \in \mathbb{R}$ such that

$$\tilde{\alpha}\exp(-\tilde{\beta}t)t^n \leq \alpha\exp(-\beta t), \quad (14)$$

where n is a natural number.

Theorem 1: Starting with any initial states $\boldsymbol{x}(0) \in \mathbb{R}^3$ and $\boldsymbol{y}(0) \in \mathbb{R}^3$, response GC system (2) equipped with controller u (12) is synchronized with drive GC system (1), and the synchronization error $\|\boldsymbol{e}\|_2$ globally exponentially converges to zero.

Proof. Substituting controller u (12) into $\dot{e}_3$ of differential equations (4) of synchronization errors yields

$$\dot{e}_3 = -\lambda^3 e_1 - 3\lambda^2 e_2 - 3\lambda e_3. \quad (15)$$

Replacing $\dot{e}_3$ of (4) with the above equation (15) yields

$$\dot{\boldsymbol{e}} = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda^3 & -3\lambda^2 & -3\lambda \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = A\boldsymbol{e},$$

with matrix $A \in \mathbb{R}^{3\times 3}$. Thus, we obtain

$$\boldsymbol{e} = \exp(At)\boldsymbol{e}(0). \quad (16)$$

The characteristic polynomial of A can be obtained as follows:

$$|sI - A| = s^3 + 3\lambda s^2 + 3\lambda^2 s + \lambda^3 = (s+\lambda)^3, \quad (17)$$

where $s \in \mathbb{R}$ is a variable of characteristic polynomial of A, and $I \in \mathbb{R}^{3\times 3}$ is the identity matrix. Evidently, the characteristic roots $s_{1,2,3} = -\lambda < 0 \in \mathbb{R}$. Thus, we obtain

$$\lim_{t\to+\infty} \|\boldsymbol{e}\|_2 = \lim_{t\to+\infty} \|\exp(At)\boldsymbol{e}(0)\|_2 = 0, \quad (18)$$

which means that the synchronization error $\|\boldsymbol{e}\|_2$ gradually converges to zero.

Based on the characteristic roots that we obtain, the Jordan canonical form $J \in \mathbb{R}^{3\times 3}$ of matrix A can be obtained as follows [7]:

$$J = T^{-1}AT = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix},$$

where $T \in \mathbb{R}^{3\times 3}$ is the nonsingular transformation matrix, and T^{-1} is the inverse matrix of T. Thus, we have the state transition matrix:

$$\exp(At) = T\exp(Jt)T^{-1} = T\exp(-\lambda t) \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} T^{-1}.$$

So, there is

$$\exp(At)\boldsymbol{e}(0) = \exp(-\lambda t)T \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} T^{-1}\boldsymbol{e}(0),$$

where $\boldsymbol{e}(0) = [e_1(0), e_2(0), e_3(0)]^{\mathrm{T}}$, and $e_i(0) = y_i(0) - x_i(0)$, with $i = 1, 2, 3$. Furthermore, we have

$$\exp(At)\boldsymbol{e}(0) = \exp(-\lambda t)[f_1(t), f_2(t), f_3(t)]^{\mathrm{T}}, \quad (19)$$

where $f_i(t) = a_i t^2 + b_i t + c_i$, with $i = 1, 2, 3$. Therefore, we have

$$\begin{aligned} |e_i| &= |\exp(-\lambda t) f_i(t)| \\ &= |\exp(-\lambda t)(a_i t^2 + b_i t + c_i)| \\ &\leq |a_i|\exp(-\lambda t)t^2 + |b_i|\exp(-\lambda t)t \\ &\quad + |c_i|\exp(-\lambda t). \end{aligned} \quad (20)$$

According to Lemma 1 and (20), we obtain

$$\begin{aligned}|e_i| &\le |a_i|\text{exp}(-\lambda t)t^2 + |b_i|\text{exp}(-\lambda t)t \\ &\quad + |c_i|\text{exp}(-\lambda t) \\ &\le \alpha_{i1}\text{exp}(-\beta_{i1}t) + \alpha_{i2}\text{exp}(-\beta_{i2}t) \\ &\quad + \alpha_{i3}\text{exp}(-\beta_{i3}t) \\ &\le 3\max_{1\le j\le 3}\{\alpha_{ij}\}\text{exp}(-\min_{1\le j\le 3}\{\beta_{ij}\}t) \\ &= \alpha_i\text{exp}(-\beta_i t),\end{aligned} \tag{21}$$

where $\alpha_{ij} > 0$, $\beta_{ij} > 0$, $\alpha_i = 3\max_{1\le j\le 3}\{\alpha_{ij}\} > 0$ and $\beta_i = \min_{1\le j\le 3}\{\beta_{ij}\} > 0$. Thus, we have

$$\begin{aligned}\|\boldsymbol{e}\|_2 &= \sqrt{e_1^2 + e_2^2 + e_3^2} \\ &\le \big((\alpha_1\exp(-\beta_1 t))^2 + (\alpha_2\exp(-\beta_2 t))^2 \\ &\quad + (\alpha_3\exp(-\beta_3 t))^2\big)^{1/2} \\ &\le \sqrt{3}\max_{1\le i\le 3}\{\alpha_i\}\text{exp}(-\min_{1\le i\le 3}\{\beta_i\}t) \\ &= \alpha\text{exp}(-\beta t),\end{aligned} \tag{22}$$

where $\alpha = \sqrt{3}\max_{1\le i\le 3}\{\alpha_i\} > 0$ and $\beta = \min_{1\le i\le 3}\{\beta_i\} > 0$. Evidently, the synchronization error $\|\boldsymbol{e}\|_2$ is of exponential convergence to zero.

Thus, we have the conclusion that response GC system (2) synthesized by controller u (12) is synchronized with drive GC system (1), and the synchronization error $\|\boldsymbol{e}\|_2$ globally exponentially converges to zero. Evidently, e_i also globally exponentially converges to zero. The proof is thus completed.

B. ZD Controller with Noise Perturbation

The performance analyses on noise-perturbed ZD controller $\tilde{u}$ (13) for synchronizing the GC system are presented in the following theorem, where $\tilde{u}$ (13) denotes the control input u with noise perturbation of response GC system (2).

Theorem 2: Starting with any initial states $\boldsymbol{x}(0) \in \mathbb{R}^3$ and $\boldsymbol{y}(0) \in \mathbb{R}^3$, response GC system (2) is synchronized relatively accurately with drive GC system (1), equipped with controller $\tilde{u}$ (13) perturbed by bounded noise $\delta(t)$, i.e., $|\delta(t)| \leqslant \delta_{\text{max}}$. Speaking in mathematics and more accurately, the steady-state error $\boldsymbol{e}$ satisfies $\lim_{t\to+\infty}\sup\|\boldsymbol{e}\|_2 < \xi$ with

$$\xi = (\delta_{\text{max}}/\lambda)\sqrt{1/\lambda^4 + 16/(\lambda^2\text{exp}(4)) + 4/\text{exp}(2)},$$

which can be decreased to be arbitrarily small by increasing $\lambda \in \mathbb{R}^+$ to be sufficiently large.

Proof. Substituting controller $\tilde{u}$ (13) into $\dot{e}_3$ of (4), we obtain

$$\dot{e}_3 = -\lambda^3 e_1 - 3\lambda^2 e_2 - 3\lambda e_3 + \delta(t). \tag{23}$$

Replacing $\dot{e}_3$ of (4) with the above equation (23) yields

$$\dot{\boldsymbol{e}} = \begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\lambda^3 & -3\lambda^2 & -3\lambda\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \delta(t)\end{bmatrix}.$$

Let $\boldsymbol{b}(t) = [0, 0, \delta(t)]^{\mathrm{T}}$. Thus, we obtain

$$\boldsymbol{e} = \exp(At)\boldsymbol{e}(0) + \int_0^t \exp(A(t-\tau))\boldsymbol{b}(\tau)d\tau. \tag{24}$$

For the matrix A, we obtain $\exp(At)$ as follows:

$$\exp(At) = \begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{21} & p_{22} & p_{23}\\ p_{31} & p_{32} & p_{33}\end{bmatrix} = [\boldsymbol{p}_1(t), \boldsymbol{p}_2(t), \boldsymbol{p}_3(t)]$$

with

$$\boldsymbol{p}_3(t) = \begin{bmatrix}p_{13}\\ p_{23}\\ p_{33}\end{bmatrix} = \begin{bmatrix}(t^2\text{exp}(-\lambda t))/2\\ (t - \lambda t^2/2)\text{exp}(-\lambda t)\\ (1 + \lambda^2 t^2/2 - 2\lambda t)\text{exp}(-\lambda t)\end{bmatrix}. \tag{25}$$

Based on $\boldsymbol{b}(t) = [0, 0, \delta(t)]^{\mathrm{T}}$ and (25), we have

$$\int_0^t \exp(A(t-\tau))\boldsymbol{b}(\tau)d\tau = \int_0^t \delta(\tau)\boldsymbol{p}_3(t-\tau)d\tau. \tag{26}$$

According to the calculus method, we have

$$\begin{aligned}&\int_0^t p_{13}(t-\tau)d\tau = \int_0^t p_{13}(\tau)d\tau \\ &= \frac{-\text{exp}(-\lambda t)(\lambda^2 t^2 + 2\lambda t + 2)}{2\lambda^3} - \left(-\frac{1}{\lambda^3}\right) \\ &= \phi_1(t) + \frac{1}{\lambda^3}, \\ &\int_0^t p_{23}(t-\tau)d\tau = \int_0^t p_{23}(\tau)d\tau \\ &= \frac{t^2\text{exp}(-\lambda t)}{2} - 0 \\ &= \phi_2(t), \\ &\int_0^t p_{33}(t-\tau)d\tau = \int_0^t p_{33}(\tau)d\tau \\ &= \frac{-t\text{exp}(-\lambda t)(\lambda t - 2)}{2} - 0 \\ &= \phi_3(t),\end{aligned} \tag{27}$$

where $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ are the primitive functions of $p_{13}(t)$, $p_{23}(t)$ and $p_{33}(t)$, respectively. Based on (27), we firstly obtain

$$\begin{aligned}&\lim_{t\to+\infty}\sup\left|\int_0^t \delta(\tau)p_{13}(t-\tau)d\tau\right| \\ &\le \lim_{t\to+\infty}\sup\int_0^t |\delta(\tau)||p_{13}(t-\tau)|d\tau \\ &\le \lim_{t\to+\infty}\int_0^t \delta_{\text{max}}|p_{13}(t-\tau)|d\tau \\ &= \delta_{\text{max}}\lim_{t\to+\infty}\int_0^t |p_{13}(\tau)|d\tau \\ &= \delta_{\text{max}}\lim_{t\to+\infty}\int_0^t p_{13}(\tau)d\tau \\ &= \delta_{\text{max}}\lim_{t\to+\infty}\left(\phi_1(t) + \frac{1}{\lambda^3}\right) \\ &= \frac{\delta_{\text{max}}}{\lambda^3}.\end{aligned} \tag{28}$$

Secondly, according to (27), we have

$$\begin{aligned}\lim_{t\to+\infty}\int_0^t p_{23}(t-\tau)d\tau &= \lim_{t\to+\infty}\int_0^t p_{23}(\tau)d\tau \\ &= \lim_{t\to+\infty}\phi_2(t)=0. \\ \lim_{t\to+\infty}\int_0^t p_{33}(t-\tau)d\tau &= \lim_{t\to+\infty}\int_0^t p_{33}(\tau)d\tau \\ &= \lim_{t\to+\infty}\phi_3(t)=0.\end{aligned} \tag{29}$$

Besides, it is evident that

$$p_{23}(t)\begin{cases}=0, & \text{if } t=0\\ >0, & \text{if } t\in(0,2/\lambda),\\ =0, & \text{if } t=2/\lambda,\\ <0, & \text{if } t\in(2/\lambda,+\infty).\end{cases}$$

$$p_{33}(t)\begin{cases}>0, & \text{if } t\in[0,(2-\sqrt{2})/\lambda),\\ =0, & \text{if } t=(2-\sqrt{2})/\lambda,\\ <0, & \text{if } t\in((2-\sqrt{2})/\lambda,(2+\sqrt{2})/\lambda),\\ =0, & \text{if } t=(2+\sqrt{2})/\lambda,\\ >0, & \text{if } t\in((2+\sqrt{2})/\lambda,+\infty).\end{cases}$$

Then, we have

$$\begin{aligned}&\lim_{t\to+\infty}\sup\left|\int_0^t\delta(\tau)p_{23}(t-\tau)d\tau\right|\\ &\le\lim_{t\to+\infty}\sup\int_0^t|\delta(\tau)||p_{23}(t-\tau)|d\tau\\ &\le\lim_{t\to+\infty}\int_0^t\delta_{\max}|p_{23}(t-\tau)|d\tau\\ &=\delta_{\max}\lim_{t\to+\infty}\int_0^t|p_{23}(\tau)|d\tau\\ &=2\delta_{\max}\int_0^{2/\lambda}p_{23}(\tau)d\tau\\ &=2\delta_{\max}\phi_2(2/\lambda)\\ &=\frac{4\delta_{\max}}{\lambda^2\exp(2)}.\end{aligned} \tag{30}$$

and

$$\begin{aligned}&\lim_{t\to+\infty}\sup\left|\int_0^t\delta(\tau)p_{33}(t-\tau)d\tau\right|\\ &\le\lim_{t\to+\infty}\int_0^t|\delta(\tau)||p_{33}(t-\tau)|d\tau\\ &\le\lim_{t\to+\infty}\int_0^t\delta_{\max}|p_{33}(t-\tau)|d\tau\\ &=\lim_{t\to+\infty}\int_0^t\delta_{\max}|p_{33}(\tau)|d\tau\\ &=-2\delta_{\max}\int_{(2-\sqrt{2})/\lambda}^{(2+\sqrt{2})/\lambda}p_{33}(\tau)d\tau\\ &=-2\delta_{\max}(\phi_3((2+\sqrt{2})/\lambda)-\phi_3((2-\sqrt{2})/\lambda))\\ &<\frac{2\delta_{\max}}{\lambda\exp(1)}.\end{aligned} \tag{31}$$

Based on (26), (28), (30) and (31), we have

$$\begin{aligned}&\lim_{t\to+\infty}\sup\left\|\int_0^t\exp(A(t-\tau))\boldsymbol{b}(\tau)d\tau\right\|_2\\ &=\lim_{t\to+\infty}\sup\left\|\int_0^t\delta(\tau)\boldsymbol{p}_3(t-\tau)d\tau\right\|_2\\ &<\left\|\left[\frac{\delta_{\max}}{\lambda^3},\frac{4\delta_{\max}}{\lambda^2\exp(2)},\frac{2\delta_{\max}}{\lambda\exp(1)}\right]^{\mathrm{T}}\right\|_2\\ &=\sqrt{\left(\frac{\delta_{\max}}{\lambda^3}\right)^2+\left(\frac{4\delta_{\max}}{\lambda^2\exp(2)}\right)^2+\left(\frac{2\delta_{\max}}{\lambda\exp(1)}\right)^2}\\ &=\frac{\delta_{\max}}{\lambda}\sqrt{\frac{1}{\lambda^4}+\frac{16}{\lambda^2\exp(4)}+\frac{4}{\exp(2)}}\\ &=\xi.\end{aligned} \tag{32}$$

Noticeably, ξ has a negative correlation with parameter λ. Finally, we obtain

$$\begin{aligned}&\lim_{t\to+\infty}\sup\|\boldsymbol{e}\|_2\\ &=\lim_{t\to+\infty}\sup\left\|\exp(At)\boldsymbol{e}(0)+\int_0^t\exp(A(t-\tau))\boldsymbol{b}(\tau)d\tau\right\|_2\\ &\le\lim_{t\to+\infty}\sup\|\exp(At)\boldsymbol{e}(0)\|_2\\ &\quad+\lim_{t\to+\infty}\sup\left\|\int_0^t\exp(A(t-\tau))\boldsymbol{b}(\tau)d\tau\right\|_2\\ &<\xi,\end{aligned}$$

which proves that an upper bound of steady-state error $\|\boldsymbol{e}\|_2$ synthesized by controller (13) is ξ. The proof is thus completed.

Corollary 1: Under the conditions of Theorem 2, we set noise $\delta(t)=\delta_{\max}$ being a constant. With the noise-perturbed controller $\tilde{u}$ (13), the synchronization error $\boldsymbol{e}$ satisfies $\lim_{t\to+\infty}\sup\|\boldsymbol{e}\|_2=\delta_{\max}/\lambda^3$, where e_1 globally converges to $\delta_{\max}/\lambda^3$, and e_2 and e_3 both globally converge to zero.

Proof. According to Theorem 1, (19) and (26), it is evident that

$$\begin{aligned}\lim_{t\to+\infty}e_1&=\lim_{t\to+\infty}\left(\exp(-\lambda t)f_1(t)+\int_0^t\delta(t)p_{13}(t-\tau)d\tau\right),\\ \lim_{t\to+\infty}e_2&=\lim_{t\to+\infty}\left(\exp(-\lambda t)f_2(t)+\int_0^t\delta(t)p_{23}(t-\tau)d\tau\right),\\ \lim_{t\to+\infty}e_3&=\lim_{t\to+\infty}\left(\exp(-\lambda t)f_3(t)+\int_0^t\delta(t)p_{33}(t-\tau)d\tau\right).\end{aligned} \tag{33}$$

We assume that the noise $\delta(t)=\delta_{\max}$ is a constant, and combine (33) with (27), thus having

$$\begin{aligned}\lim_{t\to+\infty}e_1&=0+\lim_{t\to+\infty}\int_0^t\delta_{\max}p_{13}(t-\tau)d\tau\\ &=\lim_{t\to+\infty}\delta_{\max}\left(\phi_1(t)+\frac{1}{\lambda^3}\right)=\frac{\delta_{\max}}{\lambda^3},\\ \lim_{t\to+\infty}e_2&=0+\lim_{t\to+\infty}\int_0^t\delta_{\max}p_{23}(t-\tau)d\tau\\ &=\lim_{t\to+\infty}\delta_{\max}\phi_2(t)=0,\\ \lim_{t\to+\infty}e_3&=0+\lim_{t\to+\infty}\int_0^t\delta_{\max}p_{33}(t-\tau)d\tau\\ &=\lim_{t\to+\infty}\delta_{\max}\phi_3(t)=0.\end{aligned}$$

Finally, we obtain

$$\lim_{t\to+\infty}\|\boldsymbol{e}\|_2=\lim_{t\to+\infty}\|[e_1,e_2,e_3]^{\mathrm{T}}\|_2=\frac{\delta_{\max}}{\lambda^3}.$$

Although the synchronization error $\boldsymbol{e}$ satisfies

$$\lim_{t\to+\infty}\|\boldsymbol{e}\|_2=\delta_{\max}/\lambda^3,$$

we have the achievement that response GC system (2) synthesized by controller $\tilde{u}$ (13) can also be synchronous with drive GC system (1) by increasing λ to be sufficiently large. The proof is thus completed.

III. Situation of Unequal Parameter Values

In the previous section, the performance analyses of the ZD controller without and with noise perturbation are given. In this section, the more general design and analyses of ZD controller with unequal parameter values are given. Since the theoretical analyses are similar to the previous ones, only the final theoretical results are given here.

By applying the ZD method with unequal parameter values, we obtain the following simple and direct formulas:

$$\begin{aligned}
&\dot{e}_1=-\lambda_1 e_1,\\
&z_2=\dot{e}_1+\lambda_1 e_1=e_2+\lambda_1 e_1,\\
&\dot{z}_2=-\lambda_2 z_2,\\
&z_3=e_3+(\lambda_1+\lambda_2)e_2+\lambda_1\lambda_2 e_1,\\
&\dot{z}_3=-\lambda_3 z_3.
\end{aligned}\tag{34}$$

Finally, on the basis of equation (4) and the last equation of (34), the ZD controller u is derived as follows:

$$\begin{aligned}
u=&(c-\lambda_1\lambda_2\lambda_3)e_1+(b-(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3))e_2\\
&+(a-(\lambda_1+\lambda_2+\lambda_3))e_3-(x_1+y_1)e_1.
\end{aligned}\tag{35}$$

Correspondingly, the proposed controller (35) under the pollution of the noises in practical systems becomes

$$\begin{aligned}
\tilde{u}=&(c-\lambda_1\lambda_2\lambda_3)e_1+(b-(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3))e_2\\
&+(a-(\lambda_1+\lambda_2+\lambda_3))e_3-(x_1+y_1)e_1+\delta(t).
\end{aligned}\tag{36}$$

Theorem 3: When $\lambda_1>0\in\mathbb{R}$, $\lambda_2>0\in\mathbb{R}$ and $\lambda_3>0\in\mathbb{R}$ are positive and generally unequal (including being equal as a special case), starting with any initial states $\boldsymbol{x}(0)\in\mathbb{R}^3$ and $\boldsymbol{y}(0)\in\mathbb{R}^3$, response GC system (2) equipped with controller u (35) is synchronized with drive GC system (1), and the synchronization error $\|\boldsymbol{e}\|_2$ globally exponentially converges to zero, as time t approaches infinity.

Theorem 4: When $\lambda_1>0\in\mathbb{R}$, $\lambda_2>0\in\mathbb{R}$ and $\lambda_3>0\in\mathbb{R}$ are positive and generally unequal (including being equal as a special case), starting with any initial states $\boldsymbol{x}(0)\in\mathbb{R}^3$ and $\boldsymbol{y}(0)\in\mathbb{R}^3$, response GC system (2) is relatively accurately synchronized with drive GC system (1), equipped with controller $\tilde{u}$ (13) perturbed by bounded noise $\delta(t)$, i.e., $|\delta(t)|\leqslant\delta_{\max}$. That is, the steady-state error $\boldsymbol{e}$ satisfies $\lim_{t\to+\infty}\sup\|\boldsymbol{e}\|_2<\xi$, where the bound of ξ can be controlled by $\delta_{\max}$ and $\lambda_1,\lambda_2,\lambda_3$, which can be decreased to be arbitrarily small by increasing $\lambda_1,\lambda_2,\lambda_3$ to be sufficiently large.

Corollary 2: Under the conditions of Theorem 4, we set noise $\delta(t)=\delta_{\max}$ being a constant. With the noise-perturbed controller $\tilde{u}$ (36), the synchronization error $\boldsymbol{e}$ satisfies $\lim_{t\to+\infty}\sup\|\boldsymbol{e}\|_2=\delta_{\max}/(\lambda_1\lambda_2\lambda_3)$, where e_1 globally converges to $\delta_{\max}/(\lambda_1\lambda_2\lambda_3)$, and e_2 and e_3 both globally converge to zero.

IV. Conclusion

In this paper, the problem of synchronization for GC system has been considered and investigated. By using the ZD method (i.e., via ZD steps and formulas), a suitable and effective ZD controller without or with noise perturbation to synchronize GC system has been designed and investigated. For verifying synchronization performance of the proposed controller, detailed theoretical analyses have also been provided. Kindly note that the simple design and numerical experiments of the ZD method and controllers have been provided partially in the literature [8]. All in all, the proposed ZD controller is simple, effective and robust. In light of the simpleness and feasibility of the ZD method, it has provided the possibility and potential of using the method in more practical applications.

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